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SHAPE CLASSIFICATION OF (PROJECTIVE m-SPACE)-LIKE CONTINUA

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Abstract: The Mardešić–Segal approach to shape classification of M-like continua (M a compact metric ANR) is reformulated in an algebraic setting. A functor Sh from the category of semigroups to the category of sets is defined which, when applied to $[M, M]$, the semigroup of homotopy classes maps of M into itself, yields the shape classification of M-like continua. The shape classification of all (real projective m-space)-like continua is completely solved.

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shape continuum
P-like semigroup

1. Introduction

In [1] and [2], Borsuk introduced the notion of shape for metric compacta. Shape is a classification of compacta which is coarser than homotopy type but which coincides with it on ANR's. Roughly the idea is to consider the global properties of compacta and ignore the local ones. An alternate description of shapes based on ANR-systems was given by Mardešić and Segal [7] and [8]. Furthermore in [7] all sphere like continua were classified as to shape. In [10], Segal classified all projective plane-like continua as to shape. In this paper we give the shape classification of all (real projective m-space)-like continua.

For a given compact metric ANR M, the enumeration of the shapes of M-like continua depends only on the algebraic structure of the semigroup $[M, M]$ of homotopy classes of maps of M into itself under composition. However, even if the algebraic structure of this semigroup is

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known, one is still faced with an algebraic problem which, in general, is very difficult. To facilitate our shape classification, and possibly future shape classifications, we present in Section 2 an algebraic study of shape classification of sequences in semigroups, patterned after the Mardešić–Segal formulation of shape in [7] and [8].

2. Shape classification of sequences in semigroups

Let S be a multiplicative semigroup with unit element 1. We consider sequences $s = \{s_1, s_2, ...\}$ in S.

Definition 2.1. A *map of sequences* $f: s \to t$ consists of an increasing function $f: \mathbf{Z}^+ \to \mathbf{Z}^+$, where $\mathbf{Z}^+$ denotes the positive integers, and a collection of elements $\{f_n \mid n \in \mathbf{Z}^+\}$ in S such that

$$f_n\, s_{f(n)}\, s_{f(n)+1}\, s_{f(n)+2} \cdots s_{f(n+1)-1} = t_n\, f_{n+1}.$$

The *identity map* $1_s: s \to s$ is given by $1(n) = n$, $1_n = 1$ for all n.

The *composition* of maps $f: s \to t$, $g: t \to u$ is the map $h = gf: s \to u$ defined by $h = fg$, $h_n = g_n\, f_{g(n)}$.

As an aid for thinking about maps of sequences, it might be helpful to bear in mind the following example: Let S be the semigroup of maps of a set X into itself, under composition. Interpret a sequence $\{s_1, s_2, ...\}$ in S as an inverse system

$$X \xleftarrow{s_1} X \xleftarrow{s_2} X \xleftarrow{s_3} \cdots .$$

Then a map of sequences corresponds to a map of inverse systems, as defined in [3, pp. 213, 214].

It is easily seen that composition of maps of sequences is associative and that 1_s acts as a unit.

Definition 2.2. Two maps of sequences $f, g: s \to t$ are *equivalent*, written $f \simeq g$, provided for every n in $\mathbf{Z}^+$ there exists $m \geqslant f(n), g(n)$ such that

$$f_n\, s_{f(n)}\, s_{f(n)+1}\, s_{f(n)+2} \cdots s_m = g_n\, s_{g(n)}\, s_{g(n)+1}\, s_{g(n)+2} \cdots s_m .$$

It is straightforward to check that $\simeq$ is an equivalence relation, and that if $f, f': s \to t$, $g, g': t \to u$ are such that $f \simeq f'$, $g \simeq g'$, then $gf \simeq g'f'$.

Definition 2.3. Two sequences s, s' in S are *shape-equivalent*, written $s \simeq s'$, if there exist maps of sequences $f : s \to s'$, $g : s' \to s$ such that $fg \simeq 1_{s'}$, $gf \simeq 1_s$.

It follows immediately from the above that shape-equivalence is an equivalence relation on the set of all sequences in S. Write Sh(s) for the shape-equivalence class of s, and Sh(S) for the set of all shape-equivalence classes of sequences in S.

For example, if M is a metric ANR continuum and $S = [M, M]$, the semigroup of homotopy classes of maps of M into itself under composition, and

$$(2.1) \qquad M \xleftarrow{s_1} M \xleftarrow{s_2} M \xleftarrow{s_3} \dots$$

an ANR-system, we form the sequence $s = \{[s_1],[s_2],[s_3], \dots\}$ in $[M, M]$. By [7], this establishes a one-to-one correspondence between the shape classes of compacta obtained as inverse limits of systems as in (2.1), with Sh($[M, M]$). Hence, by [6] and [5], Sh($[M, M]$) is in one-to-one correspondence with the shape classes of M-like continua. One can apply the latter result since any map of ANR continuum M into itself is homotopic to a map of M onto itself.

It is easily seen that if s and s' are sequences in S such that s' is obtained by deleting an initial segment of s, then Sh(s) = Sh(s').

If s' is obtained from s by amalgamating terms, i.e., if $s = \{s_1, s_2, \dots\}$ and $n_1 \leqslant n_2 \leqslant n_3 \leqslant \dots$, $s' = \{s_1 s_2 \dots s_{n_1}, s_{n_1+1}\, s_{n_1+2} \dots s_{n_2}, \dots\}$, it is easily proved that Sh(s') = Sh(s).

Let $f : S \to T$ be a homomorphism of semigroups with unit, i.e., f preserves multiplication and $f(1) = 1$. If $s = \{s_1, s_2, \dots\}$ is a sequence in S, define $f(s) = \{f(s_1), f(s_2), \dots\}$, a sequence in T. It is straightforward to check that if $s \simeq s'$, then $f(s) \simeq f(s')$ and so f induces a map of sets Sh(f) : Sh(S) $\to$ Sh(T), Moreover, it is easily seen that if $g : T \to U$ is another homomorphism of semigroups with unit, then Sh(gf) = Sh(g) Sh(f), and that Sh(1_S) = $1_{\mathrm{Sh}(S)}$. Thus Sh is a covariant functor from the category of semigroups with unit to the category of sets.

The semigroups S that we are concerned with also have, in addition to a unit element a zero element 0 satisfying $0s = s0 = 0$ for all $s \in S$. We will call such a semigroup a *zero–one semigroup*. A homomorphism $f : S \to T$ of zero–one semigroups will be required to satisfy $f(0) = 0$, $f(1) = 1$.

If S and T are zero–one semigroups define $S \wedge T$ to be the zero–one semigroup defined as follows: As a set

$$S \wedge T = (S\backslash\{0\}) \times (T\backslash\{0\}) \cup \{0\},$$

and multiplication is given by

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2),$$

where it is understood that $(s, t) = 0$ if either $s = 0$ or $t = 0$.

For example, if $\mathbf{Z}$ denotes the multiplicative semigroup of integers, then $\mathbf{Z}$ is isomorphic to $\{0, 1, -1\} \wedge \mathbf{Z}^*$, where $\mathbf{Z}^*$ is the multiplicative semigroup of non-negative integers.

If $s = \{s_1, s_2, ...\}$ is a sequence in S, $t = \{t_1, t_2, ...\}$ a sequence in T, define $s \wedge t$ to be the sequence in $S \wedge T$ defined by

$$s \wedge t = \{(s_1, t_1), (s_2, t_2), ...\}.$$

If $f : s \to s'$, $g : t \to t'$ are maps of sequences, define $f \wedge g : s \wedge t \to s' \wedge t'$ by

$$(f \wedge g)(n) = \max(f(n), g(n)),$$

$$(f \wedge g)_n = (f_n\, s_{f(n)}\, s_{f(n)+1}\, s_{f(n)+2} \cdots s_{(f\wedge g)(n)-1},$$
$$g_n\, t_{g(n)}\, t_{g(n)+1}\, t_{g(n)+2} \cdots t_{(f\wedge g)(n)-1}),$$

where $s_{f(n)}\, s_{f(n)+1} \cdots s_{(f\wedge g)(n)-1}$ is understood to be 1 if $(f \wedge g)(n) = f(n)$ and similarly for $x_{g(n)}\, x_{g(n+1)} \cdots x_{(f\wedge g)(n)-1}$.

Proposition 2.4. *If $f \simeq f'$ and $g \simeq g'$, then $f \wedge g \simeq f' \wedge g'$.*

Proof. For each n in $\mathbf{Z}^+$, let $m(n) \geqslant f(n), f'(n)$ be such that

$$f_n\, s_{f(n)}\, s_{f(n)+1}\, s_{f(n)+2} \cdots s_{m(n)} = f'_n\, s_{f'(n)}\, s_{f'(n)+1}\, s_{f'(n)+2} \cdots s_{m(n)}.$$

Such exist since $f \simeq f'$. Take $q(n) = \max(m(n), g(n))$. Then $q(n) \geqslant (f \wedge g)(n), (f' \wedge g)(n)$ and

$$(f \wedge g)_n (s_{(f\wedge g)(n)}, t_{(f\wedge g)(n)})(s_{(f\wedge g)(n)+1}, t_{(f\wedge g)(n)+1}) \cdots (s_{q(n)}, t_{q(n)}) =$$

$$= (f' \wedge g)_n (s_{(f'\wedge g)(n)}, t_{(f'\wedge g)(n)})$$
$$(s_{(f'\wedge g)(n)+1}, t_{(f'\wedge g)(n)+1}) \cdots (s_{q(n)}, t_{q(n)}),$$

which establishes $f \wedge g \simeq f' \wedge g$. Similarly, $f' \wedge g \simeq f' \wedge g'$.

If S is a zero–one semigroup, there is the sequence $\mathbf{0} = \{0, 0, ...\}$ in S. Denote $\mathrm{Sh}(\mathbf{0})$ by 0 in $\mathrm{Sh}(S)$, and regard $\mathrm{Sh}(S)$ as a pointed set with base point 0.

If A and B are pointed sets with base points both denoted by 0, define $A \wedge B = (A\backslash\{0\}) \times (B\backslash\{0\}) \cup \{0\}$.

Proposition 2.5. *Let S and T be zero–one semigroups. There is a natural surjection*

$$\theta : \mathrm{Sh}(S) \wedge \mathrm{Sh}(T) \to \mathrm{Sh}(S \wedge T)$$

given by $\theta(\mathrm{Sh}(s), \mathrm{Sh}(t)) = \mathrm{Sh}(s \wedge t)$.

If further $S\backslash\{0\}$ and $T\backslash\{0\}$ are closed under multiplication (i.e., no zero divisors), then θ is injective as well

Proof. By Proposition 2.4, θ is well-defined. θ is clearly natural and onto, so it remains only to show that under the added hypothesis, θ is one-to-one. We must show:

(i) If $\mathrm{Sh}(s) \neq 0 \neq \mathrm{Sh}(t)$, then $\mathrm{Sh}(s \wedge t) \neq 0$.

(ii) If $\mathrm{Sh}(s \wedge t) = \mathrm{Sh}(s' \wedge t') \neq 0$, then $\mathrm{Sh}(s) = \mathrm{Sh}(s')$ and $\mathrm{Sh}(t) = \mathrm{Sh}(t')$.

To prove (i), by deleting initial segments if necessary we can assume that s and t are sequences of non-zero terms. Hence so is $s \wedge t$. It is not hard to see that in the absence of zero-divisors, a sequence of non-zero terms cannot be equivalent to 0.

To prove (ii), we can assume that s, t, s', t' are sequences of non-zero elements. The result now follows from the following easily verifiable facts:

Any map $h : s \wedge t \to s' \wedge t'$, where each $h_n \neq 0$, can be decomposed as $h = f \wedge g$, where $f : s \to s'$, $g : t \to t'$ are maps of sequences, and if $f'f \wedge g'g \simeq 1_{s \wedge t}$, then $f'f \simeq 1_s, g'g \simeq 1_t$.

Proposition 2.6. *Let S be a zero–one semigroup. If $S\backslash\{0\}$ is a multiplicative group, then* $\mathrm{Sh}(S) = \{0, \mathrm{Sh}(1)\}$ *where* $1 = \{1, 1, 1, \ldots\}$.

Proof. If $\mathrm{Sh}(s) \neq 0$, we can assume, after deletion of an initial segment, that s is a sequence of non-zero terms. Say $s = \{s_1, s_2, \ldots\}$. Define $f : s \to 1$ and $g : 1 \to s$ by $f(n) = n = g(n)$ for all n, and $f_n = s_1 \ldots s_{n-1}$, $g_n = s_{n-1}^{-1} s_{n-2}^{-1} \ldots s_1^{-1}$. Then f, g are maps of sequences and $gf = 1_s$, $fg = 1_1$ and so $\mathrm{Sh}(s) = \mathrm{Sh}(1)$

Using the decomposition $Z \cong \{0, 1, -1\} \wedge Z^*$ and the fact that $\{1, -1\}$ is a group, it follows from Propositions 2.5 and 2.6 that the inclusion $Z^* \subset Z$ induces a bijection of sets $\mathrm{Sh}(Z^*) \cong \mathrm{Sh}(Z)$.

For $n > 0$, $[S^n, S^n] \cong Z$ as multiplicative semigroups. This isomorphism associates with a homotopy class of maps of S^n into itself the degree of a representative map. Hence, it follows from [7], where S^n-like continua were classified as to shape, that every sequence s in $\mathbf{Z}^*$, not $\simeq 0$ or 1, is shape-equivalent to a sequence of primes, and that two such sequences of primes are shape-equivalent if and only if it is possible to delete a finite number of terms from each so that every prime occurs the same number of times in each of the deleted sequences.

Let A be a set. The *free commutative zero–one semigroup on* A, denoted $\langle A \rangle$, consists of the elements 0, 1, and all formal finite monomials $s_1^{n_1} s_2^{n_2} \ldots s_r^{n_r}$, where $s_i \in A$, $n_i \in \mathbf{Z}^+$. The s_i commute, the law of exponents holds, but no other relations hold. Thus, for example, $\mathbf{Z}^* = \langle$set of positive primes$\rangle$.

Any bijection between A and B can be extended to a unique isomorphism between $\langle A \rangle$ and $\langle B \rangle$, which in turn induces a bijection between $\mathrm{Sh}(\langle A \rangle)$ and $\mathrm{Sh}(\langle B \rangle)$. Thus if A is any countably-infinite set, there is a bijection between $\mathrm{Sh}(\mathbf{Z}^*)$ and $\mathrm{Sh}(\langle A \rangle)$. Thus every sequence in $\langle A \rangle$ not equivalent to 0 or 1 is shape-equivalent to a sequence in A, and two such sequences in A are shape-equivalent if and only if it is possible to delete a finite number of terms from each so that every element of A occurs the same number of times in each of the deleted sequences.

Let S be a zero–one semigroup. A *nilpotent ideal* N in S is a nonempty subset of S such that:

(1) There exists a positive integer n such that all n-fold products of elements of N are 0.

(2) If $u \in N$ and $s \in S$, then $u\,s$ and $s\,u$ are in N.

Proposition 2.7. *Let S be a zero–one semigroup, N a nilpotent ideal in S. Let $T = (S \setminus N) \cup \{0\}$ and suppose T is closed under multiplication (so T itself is a zero–one semigroup). Then the inclusion $i : T \subset S$ induces a bijection $\mathrm{Sh}(i) : \mathrm{Sh}(T) \to \mathrm{Sh}(S)$.*

Proof. Define $p : S \to T$ by

$$p(s) = \begin{cases} s & \text{if } s \in T, \\ 0 & \text{if } s \in N. \end{cases}$$

Then p is a homomorphism of zero–one semigroups and $pi = 1_T$. Hence, by the functoriality of Sh, $\mathrm{Sh}(i)$ is one-to-one.

If a sequence s in S contains infinitely many elements of N, by amalgamating terms we obtain 0 since N is a nilpotent ideal, and so $s \simeq 0$.

Hence if $s \not\to 0$, after deletion of an initial segment of s we obtain a sequence in T, and so $\Sigma n(i)$ is onto.

3 Solenoidal projective m-spaces and classification of P^m-like continua

P^m, $m \geqslant 2$, denotes real projective m-space From Olum's [9] homotopy classification theorems, it follows that the semigroup $[P^m, P^m]$ has the following structure:

Case (1): m odd. Then $[P^m, P^m]$ is isomorphic to $\mathbf{Z}$, the multiplicative semigroup of integers. The isomorphism associates with a homotopy class of maps the degree of any map in that class.

Case (2): m even. Maps of P^m into itself are divided into two classes: Those which induce the identity homomorphism on the fundamental group, and those which induce the zero homomorphism. If f_1 and f_2 both induce the zero homomorphism, then $f_2 f_1$ is homotopically trivial, for consider the diagram

$$\begin{tikzcd}
S^m \arrow[d, "h"'] & S^m \arrow[d, "h"] & \\
P^m & P^m \arrow[l, "f_1"] \arrow[ul, "\tilde{f}_1"'] & P^m \arrow[l, "f_2"] \arrow[ul, "\tilde{f}_2"']
\end{tikzcd}$$

where $h : S^m \to P^m$ is the usual double covering map and $\tilde{f}_i$ is a lifting of f_i which exists by the Lifting Map Theorem [4, p. 89]. Applying cohomology with integer coefficients, the homomorphism

$$(\tilde{f}_1\, h)^* = h^* \tilde{f}_1^* : H^m(S^m) \to H^m(S^m)$$

is the zero homomorphism since it factors through $H^m(P^m) = \mathbf{Z}_2$. Therefore $\tilde{f}_1 h$ is of degree 0, and hence homotopically trivial since the degree of a map of S^m into itself classifies it up to homotopy. Consequently, the set N of homotopy classes of maps of P^m into itself, m even, which induce the zero homomorphism of the fundamental group, is a nilpotent ideal in $[P^m, P^m]$, in the sense of Section 2.

A map $f : P^m \to P^m$, m even, inducing the identity homomorphism on the fundamental group is classified, up to homotopy, by the absolute value of its twisted degree $\deg f$, where the twisted degree can be described in terms of the induced map in cohomology with twisted integer coefficients. The twisted degree behaves like the usual oriented degree with respect to products, i.e., $\deg(fg) = \deg f \deg g$, and precisely all the odd degrees occur. Thus when m is even, $([P^m, P^m] \setminus N) \cup \{0\}$ is isomorphic to the multiplicative semigroup of odd positive integers, together with 0, under the isomorphism which associates with a homotopy class of maps the absolute value of the twisted degree of a representative map. Thus, in the terminology of Section 2, $([P^m, P^m]\backslash N) \cup \{0\}$ is isomorphic to ⟨set of odd primes⟩ = free commutative zero–one semigroup on the set of odd primes.

We now describe a new class of P^m-like continua called solenoidal projective m-spaces. These play the same role in the shape classification of P^m-like continua that $(m-1)$-fold suspensions of solenoids did for S^m-like continua.

Case (1): m even. For each positive odd prime q, we construct a map f of P^m into itself of twisted degree q as follows: Consider S^1 as the space of complex numbers of absolute value 1. Let $g : S^1 \to S^1$ be the map which sends x to x^q. It induces a suspension map $\Sigma^{m-1} g : S^m \to S^m$ of degree q. Let $h : S^m \to P^m$ be the map which identifies antipodal points. Now f is well defined by $fh = h\,\Sigma^{m-1} g$. Let $Q = \{q_1, q_2, ...\}$ be a sequence of positive odd primes. Form the inverse sequence $P^m{}_Q = \{X_n, p_{n\,n+1}\}$ where each X_n is a copy of P^m and the map $p_{n\,n+1} : X_{n+1} \to X_n$ is the map of twisted degree q_n constructed as above. The *solenoidal projective m-space* P^m_Q is the inverse limit of $P^m{}_Q$.

Case (2): m odd. For each positive odd prime q, we construct a map f of P^m into itself of degree q as in Case (1). For $q = 2$, the composition $P^m \xrightarrow{g} R^m \cup \{\infty\} \xrightarrow{\sigma} S^m \xrightarrow{h} P^m$ has degree 2, where $R^m \cup \{\infty\}$ denotes the one-point compactification of R^m, σ is the homeomorphism inverse to stereographic projection, and g is determined by

$$gh(r_1, ..., r_{m+1}) = \begin{cases} (r_1/r_{m+1}, ..., r_m/r_{m+1}) & \text{if } r_{m+1} \neq 0 \\ \infty & \text{if } r_{m+1} = 0\,. \end{cases}$$

Let $Q = \{q_1, q_2, ...\}$ be a sequence of positive primes. Form the inverse sequence $P^m{}_Q = \{X_n, p'_{n\,n+1}\}$, where each X_n is a copy of P^m and the map $p_{n\,n+1} : X_{n+1} \to X_n$ is the map of degree q_n constructed as above. We define the *Solenoidal projective m-space* P^m_Q to be the inverse limit of $P^m{}_Q$.

Two sequences of positive primes Q and Q' are said to be *equivalent*, written $Q \sim Q'$, provided that it is possible to delete a finite number of terms from each so that every prime occurs the same number of times in each of the deleted sequences.

Theorem 3.1. *Let* P^m_Q *and* $P^m_{Q'}$ *be two solenoidal projective m-spaces. Then* $\mathrm{Sh}(P^m_Q) = \mathrm{Sh}(P^m_{Q'})$ *if and only if* $Q \sim Q'$.

Proof. This is immediate from the structures of $[P^m, P^m]$ described above, and Propositions 2.4, 2.5, 2.6 and the examples following Proposition 2.5.

A metric continuum X is said to be P^m*-like* provided for each $\epsilon > 0$ there is a mapping $f_\epsilon : X \to P^m$ onto P^m such that diam $f_\epsilon^{-1}(y) < \epsilon$ for any $y \in P^m$.

Theorem 3.2. *Every* P^m*-like continuum X has the shape of a point,* P^m *or* P^m_Q.

Proof. Since X is P^m-like, it admits an inverse sequence expansion $\{X_n, p_{n\,n+1}\}$, where all the X_n are copies of P^m (see [6]). Thus the shape classes of P^m-like continua are in one-to-one correspondence with $\mathrm{Sh}([P^m, P^m])$. The element 0 in $\mathrm{Sh}([P^m, P^m])$ yields a point, the element $\mathrm{Sh}(1)$ yields P^m, and by the structures of $[P^m, P^m]$ above, the definition of the P^m_Q, and the results of Section 2, any other P^m-like X has the shape of P^m_Q for some Q.

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